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Castaing's instability in a trapped ultra-cold gas

J.N. Fuchs[a], O. Prévoté[b], and D.M. Gangardt[c]

Laboratoire Kastler Brossel, Département de Physique de l'ENS, 24 rue Lhomond, 75005 Paris, France

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Abstract. We consider a trapped ultra-cold gas of (non-condensed) bosons with two internal states (described by a pseudo spin) and study the stability of a longitudinal pseudo spin polarization gradient. For this purpose, we numerically solve a kinetic equation corresponding to a situation close to the experiment at JILA [1]. It shows the presence of Castaing's instability of transverse spin polarization fluctuations at long wavelengths. This phenomenon could be used to create spontaneous transverse spin waves.

PACS. 03.75.Nt Other Bose-Einstein condensation phenomena – 51.10.+y Kinetic and transport theory of gases – 75.30.Ds Spin waves

1 Introduction

Recent experiments [1] with a trapped ultra-cold (non condensed) Bose gas with two internal states have shown the existence of interesting relative population dynamics. Theoretical studies [2,3] making use of a pseudo spin description for the internal states have explained this phenomenon as pseudo spin oscillations due to the "identical spin rotation effect" (ISRE) [4] which appears when the temperature is low enough for the binary collisions to be in the quantum regime. Alternatively, this mechanism can be understood as a "spin mean-field" [5]. In a polarized system, this implies the existence of low energy excitations of the transverse spin polarization, named spin waves. The prediction of spin waves in dilute gases [5,6] as well as their observations in H↓ and helium [7] goes back to the 1980's. Shortly later, Castaing [8] showed that a strong gradient of longitudinal spin polarization is unstable with respect to transverse fluctuations. His study focused on homogeneous polarized ^{3}He gas and assumed that the spin waves were in the hydrodynamic regime. The purpose of this article is to provide a quantitative study of the existence of Castaing's instability in an inhomogeneous system relevant to the experimental situation at JILA [1]. In this experiment, neither the hydrodynamic nor the collisionless regime for the spin oscillations are reached. This work is motivated by the recent contribution of Kuklov and Meyerovich [9], who were the first to suggest and study the existence of Castaing's instability in this context. For completeness, one should be aware of the numerical studies of Castaing's instability in polarized Fermi fluids, which are based on the solution of the Leggett equations rather than the full kinetic equation [10].

[a] e-mail: fuchs@lkb.ens.fr

[b] *Present address:* Université de Cergy-Pontoise, Département de Physique, 5 mall Gay Lussac, Neuville-sur-Oise, 95031 Cergy-Pontoise, France.
e-mail: olivier.prevote@voila.fr

[c] e-mail: gangardt@lkb.ens.fr

2 Kinetic equation; Leggett equations

The physical situation we consider is close to that of the experiment at JILA [1]: ^{87}Rb atoms (bosons) with two hyperfine states of interest (denoted by 1 and 2) are confined in an axially symmetric magnetic trap elongated in the Ox-direction. The temperature T is about twice the critical temperature for Bose-Einstein condensation, so that the gas is non-degenerate (Boltzmann gas). However, the de Broglie thermal wavelength is much larger than the scattering length so that the collisions occur in the quantum regime. It is convenient to consider the pseudo spin associated with the two hyperfine states 1 and 2 (the basis in spin space is denoted by $\{e_{\perp,1};\, e_{\perp,2};\, e_{\parallel}\}$). Initially the spin polarization is longitudinal (*i.e.* along the "effective external magnetic field") and has a strong spatial gradient. For example, on the left (resp. right) of the trap center, the cloud of atoms is mostly in state 1 (resp. 2). This might be achieved, for example, by preparing a cloud of atoms in state 1 and a cloud of atoms in state 2 separated by a sharp optical potential barrier at the center of the magnetic trap. After removal of the optical barrier, as the two clouds mix, a strong longitudinal spin polarization gradient appears in the region of overlap. If it is strong enough, one should be able to observe the appearance of a large transverse component of the spin polarization as a result of Castaing's instability.

To study this system, we write an effective one-dimensional kinetic equation [2] in terms of a local density in phase space $f(x,p,t)$ and (pseudo) spin density $\mathbf{M}(x,p,t)$, where x is the position, p the momentum, and t the time. The one-dimensional description for an elongated system is justified by the time scale separation associated with radial and axial motions characterized by frequencies $\omega_{\rm rad}$ and $\omega_{\rm ax}$ respectively. In the experiment at JILA, $\omega_{\rm rad}/2\pi = 230$ Hz and $\omega_{\rm ax}/2\pi = 7$ Hz [1]. The time scales differ by more than an order of magnitude leading to an effective dynamical averaging over radial dynamics as described in references [2,3]. We define the local density n and spin polarization $\mathbf{m}$ by

$$\begin{aligned} n(x,t) &= \int \mathrm{d}p\, f(x,p,t) \\ \mathbf{m}(x,t) &= \int \mathrm{d}p\, \mathbf{M}(x,p,t). \end{aligned} \tag{1}$$

The local density of atoms in state $1,2$ is then given by $n_{1,2} = (n \mp m_\parallel)/2$, defining $m_\parallel$ as the longitudinal component of the spin polarization. We also denote $g_{ij} = 4\pi\hbar^2 a_{ij}/m$ as the coupling constants associated to the different scattering lengths a_{ij} where i and $j = 1,2$ ($a_{21} = a_{12}$ and m is the mass of the particles).

We now review the different terms entering the kinetic equation (for details, see Ref. [2]). The force acting similarly on both internal states contains three terms: the magnetic trap, the mean-field, and the Stern-Gerlach force (associated to a gradient of "effective external magnetic field" $\boldsymbol{\Omega}$, see below). The magnetic trap force is dominant, so the associated potential energy is given by

$$\frac{V_1(x)+V_2(x)}{2} = \frac{1}{2} m\omega_{\rm ax}^2 x^2 \tag{2}$$

where V_1 and V_2 are the magnetic trapping potentials acting on states 1 and 2. The differential force can be described in the pseudo spin picture by an "effective external magnetic field" $\boldsymbol{\Omega}$ which contains two contributions: one due to the differential Zeeman effect and one due to the differential mean field, so that

$$\hbar\boldsymbol{\Omega}(x) = [V_2(x) - V_1(x) + (g_{22}-g_{11})n(x)/2]\mathbf{e}_\parallel \tag{3}$$

where $\mathbf{e}_\parallel$ is a unit vector in the longitudinal direction. Following reference [1], we have assumed that $2g_{12} \simeq g_{11}+g_{22}$ for simplicity. The average value over the sample of the "effective external magnetic field" is removed by going to a uniformly rotating frame (Larmor frame). It is also crucial to consider the ISRE which manifests itself as a "molecular field" or "spin mean field" $g_{12}\,\mathbf{m}(x,t)/2$, which adds to the "effective external magnetic field" (3). Finally, the collision integral is treated in a relaxation-time approximation with a time τ of the order of the average time between collisions. The kinetic equation is written as

$$\partial_t f + \frac{p}{m}\,\partial_x f - m\omega_{\rm ax}^2 x\,\partial_p f \simeq -(f - f^{\rm eq})/\tau \tag{4}$$

$$\partial_t \mathbf{M} + \frac{p}{m}\,\partial_x \mathbf{M} - m\omega_{\rm ax}^2 x\,\partial_p \mathbf{M} - (\boldsymbol{\Omega} + \frac{g_{12}\mathbf{m}}{2\hbar}) \times \mathbf{M} \simeq -(\mathbf{M} - \mathbf{M}^{\rm eq})/\tau \tag{5}$$

where $f^{\rm eq}$ and $\mathbf{M}^{\rm eq}$ are local equilibrium phase space densities.

In the hydrodynamic regime close to local equilibrium, the above equations reduce to the counterpart of the Leggett equations [11] in the case of a Boltzmann gas [6], namely

$$\begin{aligned} &\partial_t \mathbf{m} + \partial_x \mathbf{j} = \boldsymbol{\Omega} \times \mathbf{m} \\ &\partial_t \mathbf{j} - (\boldsymbol{\Omega} + \frac{g_{12}\mathbf{m}}{2\hbar}) \times \mathbf{j} + \frac{k_{\rm B}T}{m}\partial_x \mathbf{m} + \omega_{\rm ax}^2 x \mathbf{m} \simeq -\frac{\mathbf{j}}{\tau} \end{aligned} \tag{6}$$

where $\mathbf{j}(x,t)$ is the spin polarization current along Ox. These equations are also valid in the collisionless regime close to global equilibrium [11]. We define $\mu = g_{12}n(0)\tau/2\hbar$ as the dimensionless parameter which characterizes the strength of the ISRE and denote $D = k_{\rm B}T\tau/m$ as the spin diffusion coefficient.

3 Castaing's instability

Studying polarized ^{3}He gas, Castaing [8] noticed that a sufficiently strong longitudinal spin polarization gradient is unstable against transverse long wavelength fluctuations. This result was obtained by analyzing a uniform gas ($\omega_{\rm ax} = 0$) with a uniform precession frequency (so that $\boldsymbol{\Omega} = 0$ in the Larmor frame) and by assuming a time independent spin gradient. Using the Leggett equations (6), Castaing studied the stability of this system with respect to the transverse fluctuations. He assumed a small plane wave perturbation around a slightly non-uniform stationary solution such that

$$\begin{aligned} \mathbf{m}(x,t) &= m_\parallel^0(x)\mathbf{e}_\parallel + \delta\mathbf{m}\mathrm{e}^{\mathrm{i}(kx-\omega t)} \\ \mathbf{j}(x,t) &= -D\partial_x m_\parallel^0(x)\mathbf{e}_\parallel + \delta\mathbf{j}\mathrm{e}^{\mathrm{i}(kx-\omega t)}. \end{aligned} \tag{7}$$

For circularly polarized transverse spin waves in the presence of a strong gradient of longitudinal spin polarization, one obtains the following spectrum for the hydrodynamic regime ($\partial_t \mathbf{j} \simeq 0$)

$$\omega = \frac{D}{1+(\mu m_\parallel^0/n)^2}\left(\mu m_\parallel^0/n - \mathrm{i}\right)\left(k^2 - \mu k(\partial_x m_\parallel^0)/n\right). \tag{8}$$

Here, a mode with wave vector k is unstable if the mode frequency ω has a positive imaginary part $\omega_{\rm I}$, which leads to instability when

$$k^2 < \mu|k(\partial_x m_\parallel^0)|/n. \tag{9}$$

The goal of this paper is to study Castaing's instability in a trapped ultra-cold gas. Using a numerical simulation,

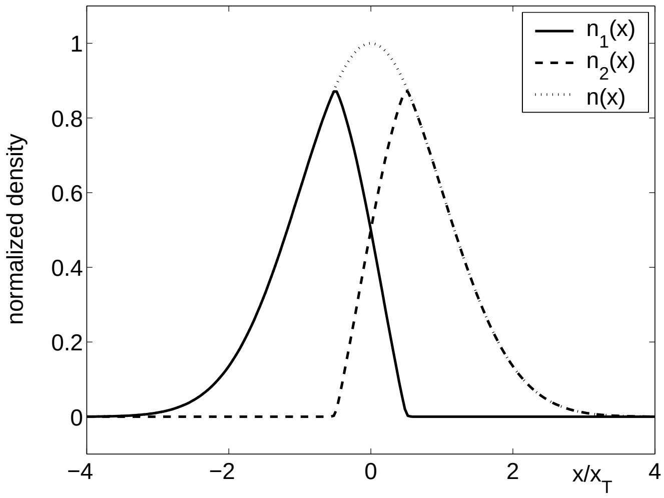

Fig. 1. Initial density of atoms in state 1 (solid line) and 2 (dashed line) giving a strong gradient of longitudinal spin polarization near the center of the trap ($\partial_x m_\parallel^0/n(0) = 2/x_\mathrm{T}$). The total density is also plotted (dotted line).

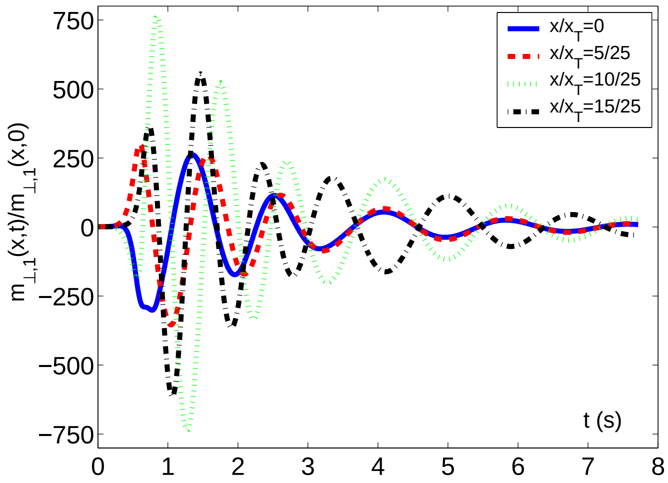

Fig. 2. Time evolution of the (normalized) transverse spin polarization $m_{\perp,1}(x,t)/m_{\perp,1}(x,0)$ at different positions in the trap: $x/x_\mathrm{T} = 0$ (solid line); 5/25 (dashed line); 10/25 (dotted line); 15/25 (dashed-dotted line). For this simulation $\mu \simeq 22$, $\omega_\mathrm{ax}\tau \simeq 0.6$, $\delta\Omega/\omega_\mathrm{ax} = 2$, $\partial_x m_\parallel^0/n(0) = 2/x_\mathrm{T}$ and $\mathcal{N} = 10^3$.

described in the next section, we show that an instability does occur. The situation we study is richer than the one considered in Castaing's original work in several aspects: (i) with the full kinetic equation (instead of Leggett equations) we can describe spin oscillations of large amplitude, (ii) we are not limited to the hydrodynamic regime (for example, regimes between hydrodynamic and collisionless are included), (iii) the longitudinal spin polarization gradient is not assumed to be constant, and (iv) we can also include a possible gradient of the precession frequency (*i.e.* the "effective external magnetic field" is position dependent).

4 Numerical simulation

The kinetic equations (4, 5) is solved numerically by propagating in time the initial distribution in a discretized phase-space using the Lax-Wendroff method (see *e.g.* [12]). We assume that the density in phase space is initially at equilibrium $f = f^\mathrm{eq} \propto \exp(-x^2/2x_\mathrm{T}^2 - p^2/2p_\mathrm{T}^2)$, with $x_\mathrm{T} = \sqrt{k_\mathrm{B}T/m\omega_\mathrm{ax}^2}$ and $p_\mathrm{T} = \sqrt{mk_\mathrm{B}T}$. As the equilibrium distribution does not evolve in time under the action of (4), the local density is constant and we concentrate on the evolution of $\mathbf{M}$ described by the kinetic equation (5). The initial spin density distribution $\mathbf{M}$ is taken as the product of the Maxwell-Boltzmann equilibrium distribution along the longitudinal axis and a function which equals -1 to the left of the trap center ($x < -x_\mathrm{T}$), $+1$ to the right ($x > x_\mathrm{T}$) and has a constant positive slope through the center ($-x_\mathrm{T} < x < x_\mathrm{T}$). The resulting initial density of atoms in state 1 and 2 is shown in Figure 1. We introduce a small initial transverse perturbation in order to start the instability. Without loss of generality, the perturbation can be written as a plane wave (with spatial frequency of order $1/x_\mathrm{T}$) multiplied by the Maxwell-Boltzmann equilibrium distribution and divided by a number $\mathcal{N} \gg 1$ (typically between 10^3 and 10^6):

$$M_{\perp,1}(x,p) = f^\mathrm{eq}(x,p)\cos(\pi x/x_\mathrm{T})/\mathcal{N}. \qquad (10)$$

Parameters used in the simulation are taken from reference [1]. The axial trapping frequency is $\omega_\mathrm{ax}/2\pi = 7$ Hz and the time between collisions $\tau \sim 10$ ms, so that $\omega_\mathrm{ax}\tau \sim 0.5$. The "effective external magnetic field" is taken to be an inverted Gaussian of depth $\delta\Omega$ ($|\delta\Omega|/\omega_\mathrm{ax}$ is varied between 0 and 2) and half-width x_T. The density at the center of the trap is $n(0) = 1.8 \times 10^{19}$ m^{-3}. The initial spin polarization gradient near the center of the trap is varied and typically $|\partial_x m_\parallel|/n(0) \sim 2/x_\mathrm{T}$. For ^{87}Rb atoms, with $a_{12} = 5.2\times10^{-9}$ m and $T = 0.6\,\mu$K, the ISRE parameter is $\mu \sim 5$. With these parameters, an instability can already be observed in our simulations but is probably too weak to be detected experimentally. To enhance the effect, we take a scattering length 5 times smaller ($a_{12} \to a_{12}/5$) and an axial trap frequency 20 times smaller ($\omega_\mathrm{ax} \to \omega_\mathrm{ax}/20$), keeping a constant density at the center of the trap. We discuss these matters further in the next section.

As the phase space is modeled by a discrete grid with finite size, it is important to distinguish between a physical instability and a numerical one. This can be easily done as only the latter will depend on the grid spacing. One can always get rid of a numerical instability by choosing a sufficiently tight grid.

Results of the simulation are shown in Figures 2 and 3. The time evolution of the transverse spin polarization is plotted in Figure 2 for different positions near the center of the trap, where the longitudinal spin polarization gradient is most pronounced. The instability is clearly visible, owing to a large enhancement of the transverse spin polarization by a large factor comparable to $\mathcal{N}$. Figure 3 shows the logarithm of the time evolution of the (normalized) transverse spin polarization at the center of the trap. In this representation, one can clearly distinguish the

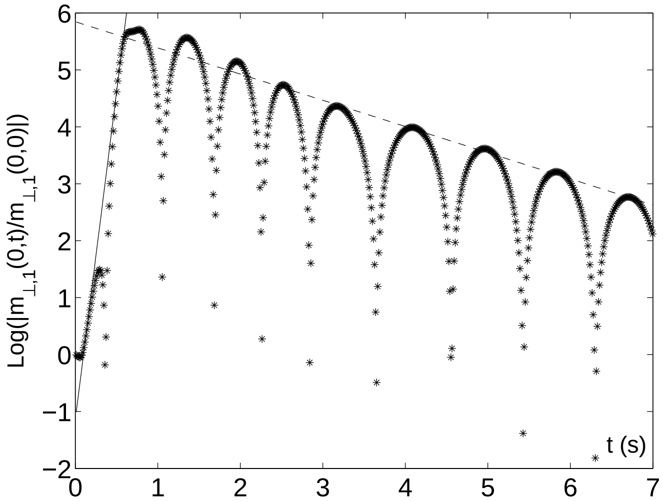

Fig. 3. Time evolution of the logarithm of the (normalized) transverse spin polarization $m_{\perp,1}(0,t)/m_{\perp,1}(0,0)$ at the center of the trap. The two lines are plotted to visualize the exponential rise of the instability (full line) and then the exponential decay of the spin wave (dashed line). The parameters have the same values than for the simulation of Figure 2.

exponential rise of the envelope (after a short delay of order of the time between collisions) and its subsequent exponential decay.

5 Discussion

The initial rise of the transverse spin polarization shows that an instability indeed occurs. As already stated, the results presented here were obtained for a scattering length a_{12} and trap frequency $\omega_{\rm ax}$ that were smaller than in the current experiment at JILA [1] (by factors of 5 and 20 respectively). This was done to enhance the effect of the "spin mean-field" (to favor the instability) by increasing the ISRE parameter μ[1]. As will be seen below, we have to keep $\omega_{\rm ax}\tau \lesssim 1$, which implies scaling the axial trap frequency with the square of the scattering length since $\tau^{-1} \propto a_{12}^2$. This procedure for enhancing the ISRE insures that the gas remains non-degenerate.

The phenomenon observed in Figures 2 and 3 may be broken down into the following four steps.

(I) During a time of the order of the time between collisions $\tau \sim 250$ ms a hydrodynamic description is not valid (the spin polarization current, which is a fast variable, has not reached its stationary value yet). Figures 2 and 3 show that the transverse spin polarization does not evolve significantly.

(II) Then, as shown by the solid line in Figure 3, the envelope of the transverse spin polarization rises exponentially (the coefficient in the exponential $\omega_{\rm I}$ is almost constant in time). This is a characteristic of an instability. The value of $\omega_{\rm I}$ (as compared to the formula obtained by Castaing, see Eq. (8)) is modified by the presence of the trap. An estimate of the time needed for the instability to develop (see Ref. [9]) is given by $t_{\rm inst} \sim L_m^2/D \simeq 1/\omega_{\rm ax}^2\tau \sim 1$ s (where L_m is the characteristic size of the longitudinal spin polarization gradient; $L_m \simeq x_{\rm T}$ in a typical simulation). It reproduces correctly the order of magnitude of the observed maximum of transverse spin polarization in the simulation. The instability can develop only if the spin polarization current has reached its stationary value, so that $\tau < t_{\rm inst}$ which implies $\omega_{\rm ax}\tau \lesssim 1$.

(III) The imaginary part of the frequency $\omega_{\rm I}$ varies slowly in time because the gradient of longitudinal spin polarization is not constant but decays as a result of both spin diffusion and the presence of the trap. The change of sign of $\omega_{\rm I}$ marks the end of the exponential grow. Once $\omega_{\rm I}$ is negative, the transverse spin polarization decays as an ordinary damped spin wave.

(IV) The longitudinal spin polarization gradient finally decays on a time scale of the order of the diffusion time $t_{\rm diff} \sim L^2/D \sim 4$ s where L, is the size of the cloud ($L/x_{\rm T} \sim 2$). As emphasized by Kuklov and Meyerovich [9], if $L > L_m$, the instability develops faster than the longitudinal spin polarization relaxes, in accordance with the results of our simulation. Once the longitudinal spin polarization is zero, $\omega_{\rm I}$ ceases to evolve in time, and consequently the envelope of the transverse spin wave decreases exponentially (dashed line in Fig. 3). This happens for $t \gtrsim t_{\rm diff}$. The frequency of the spin wave is not completely determined by the real part of the mode frequency in equation (8), as the gradient of external precession frequency, $\partial\boldsymbol{\Omega}$, and the presence of the trap are not taken into account. Actually, its order of magnitude (at the center of the trap) is given by the external precession frequency $\delta\Omega = 2\omega_{\rm ax} \simeq 2\pi\times 0.7$ Hz. We have checked that when $\delta\Omega/\omega_{\rm ax} = 0$, the transverse spin polarization at the center of the trap does not oscillate.

The criterion for an instability (see Eq. (9)) is qualitatively verified at the center of the trap if one takes into account the fact that wave vectors k are limited by the presence of the trap. This forces $k > 2\pi/L$ and the criterion becomes

$$\mu\frac{|\partial_x m_\parallel^0|}{n(0)} > \frac{2\pi}{L} \sim \frac{\pi}{x_{\rm T}}. \qquad (11)$$

Using an initial longitudinal spin polarization gradient $\partial_x m_\parallel^0/n(0) = 2/x_{\rm T}$ and $L \sim 2x_{\rm T}$ implies that the ISRE parameter μ should be larger than ~ 2. In the simulation, we found an instability threshold at $\mu \simeq 4$. Obtaining a significant instability requires a much larger value of the ISRE parameter.

We now discuss the relevance of this criterion for one of the experiments done at JILA (see the first article of Ref. [1]), where spatial separation of the two internal states was observed as a result of an initial $\pi/2$ rf pulse. The maximum longitudinal spin polarization occured at the maximum of spin state separation and can be estimated as $|\partial_x m_\parallel|/n(0) \sim 1/x_{\rm T}$ (at $x \sim \pm x_{\rm T}/2$).

[1] This could be done experimentally by using a Feshbach resonance to decrease the scattering length, as $\mu \sim \lambda_{\rm T}/a_{12}$ where $\lambda_{\rm T}$ is the de Broglie thermal wavelength.

Since the value of the ISRE parameter is $\mu \sim 5$, our numerically obtained criterion ($\mu|\partial_x m_\parallel^0|/n(0) \gtrsim 8/x_\mathrm{T}$) predicts no instability. Whether it is experimentally feasible in practice to reach transient total (or nearly total) separation of the two spin states, so that, when re-mixing, the longitudinal spin polarization gradient would be strong enough for Castaing's instability to develop, is not clear. A very strong initial gradient of longitudinal spin polarization with $\mu \sim 5$ may not be enough to start the instability because the spin diffusion is very efficient in decreasing a strong gradient. One would have to maintain a strong longitudinal spin polarization gradient in order to create an instability with the current experimental value of μ.

6 Conclusion

Our numerical simulation confirms the possibility of observing Castaing's instability in a trapped ultra-cold gas with two internal states, as proposed by Kuklov and Meyerovich [9]. We solved a one dimensional kinetic equation numerically, and were therefore able to include effects that are beyond the usual treatment of Castaing's instability in terms of a small amplitude hydrodynamic description. We argue that Castaing's instability was probably not relevant for the previous experiments done at JILA [1] as both the ISRE parameter μ and the longitudinal spin polarization gradient were too small. This does not preclude observation of the instability in future experiments if relevant parameters like the trapping frequency and the scattering length are chosen appropriately. We suggest the use of Castaing's instability as a way of creating spontaneous transverse spin waves as a result of a strong initial longitudinal spin polarization gradient. Our calculations are also valid for a non-degenerate gas of fermions (see Ref. [2]), where similar effects could be observed, in a case where $g_{11} = g_{22} = 0$ and the "spin mean-field" changes sign.

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